

Instantaneous Reflection and Transmission Coefficients and a Special Method to Solve Wave Equation

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Abstract : People are familiar with quantum mechanical reflection and transmission coefficient. In all those cases corresponding potentials are usually assumed as of constant height and depth. For the cases of varying potential, corresponding reflection and transmission coefficients can be found out using WKB approximation method. But due to change of barrier height, reflection and transmission coefficients should be changed from point to point. Here we show the analytical expressions of the instantaneous reflection and transmission coefficients. Here as if we apply the WKB approximation at each point, so we call it as Instantaneous WKB method or IWKB method. Once we know the forward and backward wave amplitudes we can find out corresponding wave function by calculating *Eiconal*. For the case of analytically complicated potential corresponding differential equation seems to be unsolvable analytically. If the potentials are well behaved then obviously these could be replaced by functions of simple expression of same behaviour which can be integrated analytically and the equation is now possible to solve analytically.

KEY WORDS : WKB approximation, reflection and transmission coefficients, complicated analytical form, potential barrier

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I. INTRODUCTION

In the cases of quantum mechanical barrier problem reflection and transmission coefficients can be calculated. In those cases with the interaction of the potential field, one fraction of incident particle (or wave) is reflected back to the infinity (reflection coefficient) and other fraction transmits into other side or may tunnel through the barrier (transmission coefficient). If the barrier is of constant height and depth, the reflection and transmission coefficient can be easily calculated [1]. In the case of varying potential, from the far away of potential field, reflection and transmission coefficient can be calculated using WKB approximation [1,2]. In this present paper we will show how the reflection and transmission of the particle changes in the cases of varying potential point to point. Due to variation of potential, in each point particle should feel different potential field so the fraction of particle (or wave) which reflects and transmits will be changing at each point. Here we will calculate the analytical space dependent expression of reflection and transmission coefficients which indicates transmission and reflection of the particle at any arbitrary point with respect to the transmission of the particle at the immediate previous point. So the local values of incident and reflected wave amplitude can be found. With this, we shall present the solution of the differential equation (i.e. Schrödinger equation), containing that space dependent reflected and transmitted amplitude.

Secondly, in our daily problems of physics and mathematics there are several differential equations which can not be solved analytically. If we want to solve those equations we need numerical methods. If, however, the nature of the coefficients of derivative terms or inhomogeneous term namely factor, due to presence of those factors corresponding differential equation to be unsolvable, are well behaved then using the methods which will be explained here the differential equation can be solved analytically using WKB approximation method. Actual reason of non-analyticity is that corresponding factors can not be integrated analytically.

We can consider, a particle is moving in a potential field whose analytical form is complicated. Obviously corresponding Schrödinger equation will be non-analytical. Although the reflection and transmission coefficients for that

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particle can be calculated but the wave function can not be analytically calculated exactly, because for this job we need to integrate analytically the wave vector k , which includes the complicated potential.

In next section we will discuss how the space dependent reflection and transmission coefficient can be calculated for the case of varying potential. In §3, we will establish approximate analytical solution of a Schrödinger equation with the presence of potential with complicated analytical form. In §4 we will illustrate an example where our method can be applied for finding solution and reflection-transmission coefficients. In §5 we will make our conclusions.

II. METHOD TO FIND LOCAL REFLECTION AND TRANSMISSION COEFFICIENTS

We consider an one dimensional Schrödinger like differential equation as

$$\frac{d^2y}{dx^2} + k^2(x)y = 0 \quad (1)$$

where, $k^2(x) = E - V(x)$, E = total energy, $V(x)$ = potential energy. (2)

Here, the independent variable x is varying from $-\infty$ to ∞ , i.e., nothing but cartesian co-ordinate.

Using WKB method the solution of this equation is [1,2]

$$y(x) = \frac{A}{\sqrt{k}} \exp(iu) + \frac{B}{\sqrt{k}} \exp(-iu) \quad (3)$$

where,

$$u = \int k(x)dx, \quad (4)$$

A and B are constants of integration.

The reflection and transmission coefficients calculated by WKB method means the coefficients are calculated at a certain point, that means from a certain point what fraction of particle is transmitted inside and what other fraction is reflected outside. If we calculate this transmission and reflection of the particle from different points, results should be different. So the probability of transmission and reflection of the particle from point to point will be different for the case of varying potential. So the constants A and B should be changed from point to point due to change of effective potential which is felt by the particle. Here we will give an analytical space dependent expression for transmission and reflection coefficient by which one can find, how the reflection and transmission probability change from point to point for the case of varying potential.

It is assumed that far away from the potential field (at $x = \infty$), the potential barrier height is almost constant, so the reflection and transmission coefficients are almost same. In those regions we safely can choose A and B as pure constants. Again, all along the sum of transmission and reflection coefficients should be unity, so far away from the potential field the relations between A and B are,

$$A = B + c (c = \text{constant}), \quad (5)$$

$$A^2 + B^2 = k. \quad (6)$$

It is very easy to check from incident and reflected current that transmission and reflection coefficients are $\frac{A^2}{k}$ and $\frac{B^2}{k}$ respectively.

Solving these two equation we get,

$$A(x) = \frac{c}{2} + \frac{\sqrt{2k - c^2}}{2} \quad (7a)$$

$$B(x) = -\frac{c}{2} + \frac{\sqrt{2k - c^2}}{2} \quad (7b)$$

These $A(x)$ and $B(x)$ are very slowly varying functions at large x . These are indicating, at far away from the potential barrier (where potential is varying in very slow manner), how constants A and B are slowly changing. So from these slowly varying A and B , one can find out how transmission and reflection coefficients are varying at far

distance. Now, constant c can be calculated from the boundary conditions. From pure WKB approximation constant c can be calculated. From WKB method, far away from the potential field reflection and transmission coefficients and corresponding reflected and transmitted amplitude can be calculated. So constant c is immediately determined. Obviously, at $x = -\infty$ these expressions of A and B will not be valid because of simplified assumptions. At the minimum value of x , i.e., $x = -\infty$, we have to put another condition.

It is clear that the potential field, which is felt by the particle, extended in between $-\infty$ to ∞ and assumed that at $x = -\infty$ barrier height goes down to zero or constant value i.e., the potential is asymptotically flat. So the particle which reaches at $x = -\infty$, should feel free. As a result, there transmission should be 100% and corresponding reflection is zero (hereafter, this will be called as inner boundary condition). By introducing this inner boundary condition in eqn. (7) we get,

$$C(x) = c_1 + A(x) = c_1 + \frac{c}{2} + \frac{\sqrt{2k - c^2}}{2}, \quad (8a)$$

$$D(x) = c_2 + B(x) = c_2 - \frac{c}{2} + \frac{\sqrt{2k - c^2}}{2}. \quad (8b)$$

Here, constants c_1 and c_2 are introduced to modify the reflection and transmission coefficients according to the inner boundary condition.

Here, one necessary condition is the sum of reflection and transmission coefficients should be one. By modifying the coefficients, it is seen that

$$C^2(x) + D^2(x) = \left(c_1 + \frac{c}{2}\right)^2 + \left(c_2 - \frac{c}{2}\right)^2 + (c_1 + c_2)\sqrt{2k - c^2} + \frac{(2k - c^2)}{2} = h(x)[say] = \left[\frac{h(x)}{k(x)}\right] k(x). \quad (9)$$

So, it is advisable to choose the modified coefficients of the wave function as follows:

$$a(x) = \frac{C(x)}{\sqrt{h/k}} \quad (10a)$$

$$b(x) = \frac{D(x)}{\sqrt{h/k}} \quad (10b)$$

so that

$$a^2(x) + b^2(x) = k(x) \quad (11)$$

and the transmission and reflection coefficients are $\frac{a^2(x)}{k(x)}$ and $\frac{b^2(x)}{k(x)}$ respectively which are explicitly written as:

$$T(x) = \frac{\left(c_1 + \frac{c}{2}\right)}{h(x)} \left(c_1 + \frac{c}{2} + \sqrt{2k(x) - c^2}\right) + \frac{2k(x) - c^2}{4h(x)} \quad (12a)$$

$$R(x) = \frac{\left(c_2 - \frac{c}{2}\right)}{h(x)} \left(c_2 - \frac{c}{2} + \sqrt{2k(x) - c^2}\right) + \frac{2k(x) - c^2}{4h(x)}. \quad (12b)$$

Now we describe, how to determine the constants c_1 and c_2 . First of all to satisfy $B(x = -\infty) = 0$ (inner boundary condition) c_2 is determined and fixed. Then introducing this c_2 in $h(x)$ and equating the expression of modified reflection coefficient [$b^2(x)/k(x)$] and reflection coefficient valid only at $x = \infty$ [$B^2(x)/k(x)$, which is correct at far away from the black hole] we get the other constant c_1 . These expressions for transmission and reflection coefficients are valid in whole region. The space dependent coefficients of incident and reflected waves, $a(x)$ and $b(x)$ are valid for arbitrary x .

So the final form of the solution using WKB method is

$$y(x) = \frac{a(x)}{\sqrt{k}} \exp(iu) + \frac{b(x)}{\sqrt{k}} \exp(-iu). \quad (13)$$

Here one important thing is to be noted that, WKB approximation is valid only if $\frac{1}{k} \frac{dk}{dx} \ll k$. So one can calculate the space dependent reflection and transmission coefficient only if potential vary in such a manner that k satisfies the

above condition. Otherwise our method to find out space dependent amplitudes and coefficients is not valid because our method is based on WKB approximation. Equation (13) is valid at any point. It is indicating the solution and corresponding reflection and transmission coefficients for an arbitrary x as if at each point WKB approximation method is applied instantaneously. So our modified WKB method can be called as *Instantaneous WKB Method* or IWKB method [3]. For the cases, where $\frac{1}{k} \frac{dk}{dx} \ll k$ is not satisfied, the potential can be replaced by a large number of square steps and space dependent reflection and transmission coefficients can be found [3]. Here, corresponding problem is reduced to as simple quantum mechanical barrier problem with multiple steps where at each step junction the wave functions and its derivatives of two separate regions should be continuous. Detail discussion is in Mukhopadhyay & Chakrabarti [3].

III. SOLUTION OF THE EQUATION

In the previous section, although we have given the form of the solution with space dependent incident and reflected coefficients but the *Eiconal* u still yet to be determined. If the analytical form of the potential and corresponding wave vector k is well integrable analytically then u can be calculated immediately but sometimes the form of k may be complicated such that it can not be integrated analytically then u can not be evaluated unless very special cases are chosen. In this section we want to give an approximate solution of eqn. (1) using WKB (actually IWKB, as mentioned in last section) approximation even if the analytical form of k is not well integrable. As we mentioned in INTRODUCTION that if $k^2(x)$ is well behaved then we can give an analytical form of the eqn. (1) using the following trick. If the nature of the coefficients of derivative and zeroth order derivative terms are well behaved but analytically complicated then those analytical form of the coefficients can be replaced by piecewise continuous analytical function with simple form and of same behaviour as exact complicated functions. After this replacement k will be of simple form and we can integrate it to find out u . Once we know the analytical form of u , we can present the analytical solution of the equation.

In different ways we can replace the complicated functions. Any function can be expanded in terms of polynomial. It is seen that if the corresponding polynomial is of order two then the corresponding k can be integrated easily. But using one polynomial of order two, in the total range, the replacement of the complicated function can not be done. So in different regions, different coefficients of the polynomial are chosen i.e., different second order polynomials are chosen such that at the boundaries (where the polynomials change), value of the two different polynomials and its derivatives are same. In this way, by using many number of piecewise polynomials the analytically complicated well behaved functions can be replaced.

If in the differential eqn. (1), the analytical form of V and corresponding k is of non-analytical form (complicated analytical expression) then potential V can be replaced by piecewise continuous polynomial as

$$V_l(x) = a_l + b_l x + c_l x^2 \quad (14)$$

and the corresponding $k(x)$ is defined as

$$k_l^2(x) = (E - a_l) - b_l x - c_l x^2. \quad (15)$$

Integrating this we can find out analytical form of u as

$$u_l(x) = -2c_l k_l - c_l \sqrt{(E - a_l)} \log \left| \frac{k_l - \sqrt{(E - a_l)}}{k_l + \sqrt{(E - a_l)}} \right| + \text{constant} \quad (16)$$

Where index l in the coefficients of the polynomial is indicating that in different ranges of x different values of a , b , c may be chosen. This type of replacement also can be done by some other well integrable functions according to the nature of the complicated functions. In the next section we will give an example, where, to solve the differential equation with the potential of complicated analytical form, we replace the complicated function (potential) by the function of simple analytical form otherthan second order polynomials.

In this way we can give the analytical form of the solution of Schrödinger equation like equation (3) even if the analytical form of the potential is complicated. Since we have used the WKB approximation method (more clearly IWKB method), our solution only will be valid for $\frac{1}{k} \frac{dk}{dx} \ll k$. If the potential vary in such a way that above condition is not being satisfied then the solution of the Schrödinger or Schrödinger like equation is not possible by this method. For those cases we have to employ some other methods, one of them is indicated in last section (by reducing the problem to simple quantum mechanical barrier problem). Sometimes for those cases, the Schrödinger equation can be reduced into Bessel equation and solution is possible by expressing in terms of Airy functions [1].

IV. ILLUSTRATIVE EXAMPLE

Dirac equation in Kerr geometry can be separated into radial and angular parts [4-5]. Then corresponding decoupled radial equation on some transformation of independent variable can be reduced into Schrödinger like equation [5] as

$$\frac{d^2Z}{dx^2} + (\sigma^2 - V(x)) = 0. \quad (17)$$

In comparison with Schrödinger equation with $\hbar = c = G = 1$, we can say $V(x)$ is nothing but potential and σ^2 is proportional to energy of the particle. Here the analytical form of the potential is complicated as

$$V = \frac{\Delta^{\frac{1}{2}}(\lambda^2 + m_p^2 r^2)^{3/2}}{[\omega^2(\lambda^2 + m_p^2 r^2) + \lambda m_p \Delta / 2\sigma]^2} [\Delta^{\frac{1}{2}}(\lambda^2 + m_p^2 r^2)^{3/2} + ((r - M)(\lambda^2 + m_p^2 r^2) + 3m_p^2 r \Delta)] - \frac{\Delta^{\frac{3}{2}}(\lambda^2 + m_p^2 r^2)^{5/2}}{[\omega^2(\lambda^2 + m_p^2 r^2) + \lambda m_p \Delta / 2\sigma]^3} [2r(\lambda^2 + m_p^2 r^2) + 2m_p^2 \omega^2 r + \lambda m_p (r - M) / \sigma], \quad (18)$$

where a = Kerr parameter, m_p = mass of the incident particle, σ = frequency of the incident particle, M = mass of the black hole, λ = separation constant, r = radial co-ordinate, $\Delta = r^2 - 2Mr + a^2$, $\omega = r^2 + a^2 + am/\sigma$, $x = f(r)$. For details see [5].

Now, although the analytical form of the potential is complicated but for particular set of physical parameter (such as, a, σ, m_p etc.) the nature of V is well behaved. So we can replace it by suitable piece-wise continuous analytical function of simple form. We can choose it as second order polynomial as explained in previous section. But for convenience we choose here in different form.

To show the nature of the potential, we choose as an example one set of physical parameter, such as:

$a = 0.5$; $M = 1$; $m_p = 0.8$; orbital quantum number, $l = \frac{1}{2}$; azimuthal quantum number, $m = -\frac{1}{2}$; $\sigma = 0.8$; $\lambda = 0.92$ from [6].

The simple analytical form which is chosen to replace the complicated form of the potential for this set of physical parameter is given as

$$V_l(x) = a_l + b_l \exp\left(-\frac{x}{c_l}\right) \quad (19)$$

and corresponding k and eiconal u are given as

$$k_l(x) = \sqrt{(\sigma^2 - a_l) - b_l \exp(-\frac{x}{c_l})}, \quad (20)$$

$$u(x) = -2c_l k_l(x) + c_l \sqrt{(\sigma^2 - a_l)} \log \left| \frac{\sqrt{(\sigma^2 - a_l)} + k_l(x)}{\sqrt{(\sigma^2 - a_l)} - k_l(x)} \right| + \text{constant.} \quad (21)$$

The behaviour of the functions of eqn. (18) and (19) are same. In case of (19) in different ranges of x different values of a, b, c are chosen.

For these physical parameters, coefficients a, b, c of mapping function in different ranges of x are given as:

$a = 0, b = -0.187354, c = -3.75$ for $x \rightarrow \infty$ to 0,

$a = 0.603, b = 0.415646, c = 8.79$ for x 0 to 30,

$a = 0.629, b = 0.12690038, c = 26.3$ for x 30 to 109,

$a = 0.63543098, b = 0.037193439, c = 73.5$ for x 109 to 208,

$a = 0.63543098, b = 0.2228925, c = 45$ for x 208 to 310,

etc.

So finally we can say the analytical form of the solution of the Schrödinger equation is possible for this set of physical parameter. Similarly for other sets of physical parameter one can find out solution following same method (by mapping the complicated analytical form of the potential by the simple form which may be polynomial or analytical function like eqn. (19) and which are analytically integrable).

The space dependent transmission and reflection coefficients can be calculated following the method explained earlier. Now we will calculate the constants c, c_1, c_2 by imposing boundary conditions for the potential of given set of

physical parameter. Using general WKB approximation method, from the far away of the black hole (say at $x = 310$) the reflection and transmission coefficients of the particle for the potential shown in Fig. 1 can be calculated as $T = 0.299$ and $R = 0.701$. Using these values of the coefficient constant c can be calculated as

$$c = \sqrt{Tk} - \sqrt{Rk} = -0.0913 \quad (22)$$

where k is the value of wave number at $x = 310$.

Using this c when we calculate the transmitted and reflected amplitude from (7a-b) (which are valid at large distance from the black hole) obviously it will not satisfy the inner boundary condition. To satisfy the inner boundary condition first c_2 is introduced to reduce $B(x)$ to zero at $x = -\infty$ as $c_2 = -0.6765$. Then as explained in section II, by matching the reflection amplitudes of eqn. (7b) and (10b) at infinity i.e.,

$$b(x) = B(x), \quad (23)$$

c_1 can be evaluated as $c_1 = 0.1639$.

Now all the constants are known. In Fig. 2 variation of instantaneous reflection and transmission coefficients are shown. It is seen that as the potential barrier lower down the transmission probability increases and reflection probability decreases. For the cases of other set of physical parameters where the potential barrier may be of different type, using the same method constants can be calculated. In the final solution with the insertion of the value of constants it can be checked that WKB approximation is still valid except in the regions where σ^2 is close to V and this modified WKB (IWKB) solution satisfies the original differential equation if $\frac{1}{k} \frac{dk}{dx} \ll k$.

V. CONCLUSIONS

Here, we show how reflection and transmission coefficients vary for the particle moving in spatially varying potential field. Actually we solve the Quantum Mechanical barrier problem, where the barrier height changes point to point. For solution, we introduce modified WKB approximation method namely instantaneous WKB method i.e., IWKB method where at each point, instantaneously, the WKB approximation method can be applied and corresponding reflection and transmission coefficients can be calculated. Since the potential changes point to point corresponding reflection and transmission coefficients change. We also indicate how Schrödinger equation is solved analytically where coefficients of derivative terms are of complicated analytical form. It is shown that if coefficients are well behaved then the the analytical solution can be set up by suitably mapping the coefficient-functions to functions of simple form. In this way, any such kind of differential equation can have analytical form of the solution which could be useful for further study.

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FIGURE CAPTION

Fig. 1 : Variation of potential V (solid curve) with respect to radial distance x and total energy E (dashed curve) of particle.
Fig. 2 : Variation of transmission (solid curve) and reflection coefficients (dashed curve) with respect to radial distance x .



